1 Vector & Matrix Norms

A **vector norm** on $\mathbb{R}^n$ is a real map $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$ satisfying the following:

1. $\forall \vec{x} \in \mathbb{R}^n.[\vec{x} \neq \vec{0} \implies \|\vec{x}\| > 0]$.
2. $\forall \lambda \in \mathbb{R}.\forall \vec{x} \in \mathbb{R}^n.[\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|]$.
3. $\forall \vec{x}, \vec{y} \in \mathbb{R}^n.[\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|]$.

For $p > 0$ the **ℓ_p-norm** of $\vec{x}$ is defined as:

$$\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

$$\|\vec{x}\|_1 = \sum_{i=1}^n |x_i|$$

$$\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

$$\|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Any vector norms $\|\cdot\|_a, \|\cdot\|_b$ in $\mathbb{R}^n$ are **equivalent**: $\exists r, s > 0$ s.t. $\forall \vec{x} \in \mathbb{R}^n.[r\|\vec{x}\|_a \leq \|\vec{x}\|_b \leq s\|\vec{x}\|_a]$.

A **matrix norm** on $\mathbb{R}^{m\times n}$ is a real map $\|\cdot\|: \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying the following:

1. $\forall A \in \mathbb{R}^{m\times n}.[A \neq 0 \implies \|A\| > 0]$.
2. $\forall \lambda \in \mathbb{R}.\forall A \in \mathbb{R}^{m\times n}.[\|\lambda A\| = |\lambda|\|A\|]$.
3. $\forall A, B \in \mathbb{R}^{m\times n}.[\|A+B\| \leq \|A\| + \|B\|]$.

In a **sub-multiplicative** matrix norm, $\forall A \in \mathbb{R}^{m\times n}.\forall B \in \mathbb{R}^{n\times p}.[\|AB\| \leq \|A\|\|B\|]$.

We define **ℓ_p-norms** of a matrix as:

$$\|A\|_1 = \max_j \|a_j\|_1$$

$$\|A\|_2 = \sigma_1(A)$$

$$\|A\|_\infty = \|A^T\|_1$$

$$\|A\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2}$$

A matrix norm on $\mathbb{R}^{m\times n}$ is **consistent** on vector norms $\|\cdot\|_a$ on $\mathbb{R}^n$ and $\|\cdot\|_b$ on $\mathbb{R}^m$ iff $\forall A \in \mathbb{R}^{m\times n}.\forall \vec{x} \in \mathbb{R}^n.[\|A\vec{x}\|_b \leq \|A\|\|\vec{x}\|_a]$. If $a = b$ then they are **compatible**. For a vector norm, the **subordinate** matrix norm is $\forall A \in \mathbb{R}^{m\times n}, \|A\| = \max\{\|A\vec{x}\| : x \in \mathbb{R}^n, \|\vec{x}\| = 1\}$. *A vector norm is compatible with its subordinate matrix norm.*

2 Linear Maps on $\mathbb{C}^n$

The **standard inner product** is defined as $\forall \vec{u}, \vec{v} \in \mathbb{C}^n.\langle u, v\rangle = \overline{u}^T v = \sum_{i=1}^n \overline{u_i} v_i$.

3 Least Square Method

> The **orthogonal projection** $\pi_U : \mathbb{R}^m \to \mathbb{R}^m$ is $\vec{v} \mapsto \pi_U(\vec{v}) = U(U^T U)^{-1} U^T \vec{v}$. $\operatorname{im} A \perp \ker A^T$, hence $\forall \vec{x} \in \mathbb{R}^m$, there exist unique $\vec{x_i} \in \operatorname{im} A, \vec{x_k} \in \ker A$ s.t. $\vec{x} = \vec{x_i} + \vec{x_k}$.

Suppose $\forall \vec{x} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has no solution (i.e. $\vec{b} \notin \operatorname{im} A$). The **Least Square Method** finds $x \in \mathbb{R}^n$ s.t. $\|A\vec{x} - \vec{b}\|_2$ is *minimized*. This happens iff $\|A\vec{x} - \vec{b_i}\|_2 = 0$, or $A\vec{x} = \vec{b_i}$.

$A^T A\vec{x} = A^T\vec{b}$ is the **normal equation**, whose solution also solves the least square problem. To find the best **affine map** (straight line) $Y = mX + c$ we solve

$$A^T A \begin{bmatrix} c \\ m \end{bmatrix} = A^T Y \text{ where } A = \begin{bmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}.$$

3.1 Linear Regression

Suppose we have a set of points $(y_i, \vec{a_i})$ where $y_i \in \mathbb{R}$, $\vec{a_i} \in \mathbb{R}^n$. We want to find a *model of best fit* with params $s_0 \in \mathbb{R}, \vec{s} \in \mathbb{R}^n$ s.t. the sum of errors squared $\sum_{i=1}^m (s_0 + \vec{s}\cdot\vec{a_i} - y_i)^2$ is minimized: $\forall i.[s_0 + \vec{s}\cdot\vec{a_i} \approx y_i]$. To solve with least squares, let:

$$A = \begin{bmatrix} 1 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{m1} & \cdots & a_{mn} \end{bmatrix}, \vec{z} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix} \in \mathbb{R}^{n+1}$$

$$\sum_{i=1}^m (s_0 + \vec{s}\cdot\vec{a_i} - y_i)^2 = \|A\vec{z} - \vec{y}\|_2^2$$

4 Spectral Decomposition

> A matrix $Q \in \mathbb{R}^{n\times n}$ is **orthogonal** iff it is invertible and $Q^{-1} = Q^T$. This implies $|\det Q| = 1$, & $\forall$ eigenvalues λ of Q, $|\lambda| = 1$.
>
> A matrix $A \in \mathbb{R}^{n\times n}$ is **symmetric** iff $A^T = A$. Then, all eigenvalues of A are *real*, and their *algebraic and geometric multiplicities are equal*. Also, any eigenvectors with different eigenvalues are orthogonal.

Any **symmetric** matrix $A \in \mathbb{R}^{n\times n}$ can be diagonalized as $A = QDQ^T = QDQ^{-1}$, where Q is *orthogonal* and D is *diagonal* matrix of not necessarily distinct eigenvalues. To find the SD of A:

1. Find characteristic polynomial of A, solve to find eigenvalues λ_i of A.
2. For each distinct λ_i, find corresponding eigenspace E_{λ_i}.
3. For each E_{λ_i}, find orthonormal basis $(\vec{v_{\lambda_i 1}}, \vec{v_{\lambda_i 2}}, \cdots, \vec{v_{\lambda_i \dim E_{\lambda_i}}})$.
4. The bases $\vec{v_{\lambda_i}}$ are columns of Q and eigenvectors of A. Done!

5 Singular Value Decomposition

> A **symmetric** matrix $A \in \mathbb{R}^{n\times n}$ is **positive semi-definite** (*non-negative eigenvalues*) iff $\forall \vec{x} \in \mathbb{R}^n.\vec{x}^T A\vec{x} \geq 0$, and **positive definite** (*positive eigenvalues*) iff $\forall \vec{x} \in \mathbb{R}^n - \{0\}.\vec{x}^T A\vec{x} > 0$.
>
> For any $A \in \mathbb{R}^{m\times n}$, $A^T A \in \mathbb{R}^{n\times n}$ and $AA^T \in \mathbb{R}^{m\times m}$ are *symmetric* & *positive semi-definite*.

For any $A \in \mathbb{R}^{m\times n}$, the **Singular Value Decomposition** of A is $A = USV^T$, where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are **orthogonal**, and $S = \operatorname{diag}(\sigma_1, \cdots, \sigma_p)$ and $p = \min(m, n)$ where $\sigma_1 \geq \cdots \geq \sigma_p \geq 0$ are the **singular values** of A. This gives rise to properties:

- The rank $r = \operatorname{rk}(A)$ is equal to the number of positive singular values in S: $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_p = 0$.
- $\|A\|_2 = \sigma_1$.
- The positive singular values of A are the positive square roots of the eigenvalues of AA^T or $A^T A$.
- $U^T U = I$.

To compute SVD of A, we pick the smaller of $A^T A$ and AA^T and then:

1. Solve $\det(A^T A - \lambda I) = 0$ to get $\lambda_1, \cdots$.
2. Take sqrts to get $\sigma_1, \cdots$. Skip $\lambda_i = 0$.
3. Find v_i by solving $(A^T A - \lambda_i I)v_i = \vec{0}$.
4. v_i are cols of V. *MUST BE orthonormal.*
5. Construct S from σ_i (same shape as A).
6. $\forall \sigma_i \neq 0$, $u_i = \frac{1}{\sigma_i} A v_i$.
7. u_i are cols of U. *If cols are missing use gram schmidt or cross product.*

> ***If you pick AA^T then calc U before V in the same way.*** Also:
>
> - $\operatorname{rk}(A) = \text{num of } \sigma_i \neq 0$.
> - $\|A\|_2 = \sigma_{\max}(A)$.
> - $\operatorname{im}(A) = \operatorname{span}\{u_1, \cdots u_r\}$, $r = \operatorname{rk}(A)$.

5.1 Principal Component Analysis

Assume $A \in \mathbb{R}^{m\times n}$ represents m samples of n-dim data. The **principal axes** of A are the cols of V, and the cols of US (i.e. $\sigma_i u_i$) are the **principal components** of A.

> $w^T A^T A w$ is maximised $V^T w = e_1$, $w = v_1$. Assuming $\|w\|_2 = 1$.

$A = \sum_{i=1}^r \sigma_i u_i v_i^T$. Then to max $\|Ax\|_2$, we need to align $x = v_1$ (since v_1 corresponds to largest σ_r. This maxes the contributions.

6 Generalized Eigenvectors

For $A \in \mathbb{R}^{n\times n}$, non-zero $v \in \mathbb{C}^n$ is a **generalized eigenvector of rank m (v^m)** with assoc eigenvalue $\lambda \in \mathbb{C}$ iff $(A - \lambda I)^m v = 0$ and $(A - \lambda I)^{m-1} v \neq 0$.

If an eigenvalue λ has algebraic multiplicity k, there are k linearly independent generalized eigenvectors associated with λ. $A \in \mathbb{R}^{n\times n}$ has n linearly independent generalized eigenvectors - $\exists$ a basis of $\mathbb{C}^n$ of generalized eigenvectors of A.

> They always solve $(A - \lambda_i I)v_g = v_i$.

6.1 Jordan Normal Form (JNF)

A matrix is in JNF if it is in the form:

$$\begin{bmatrix} J_{k_1}(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{k_n}(\lambda_n) \end{bmatrix}$$

where $J_{k_i}(\lambda_i)$ is a **jordan block** of size k_i:

$$\begin{bmatrix} \lambda_i & 1 & \cdots \\ 0 & \lambda_i & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

The algebraic multiplicity of λ is the sum of the sizes of blocks with λ on the diagonal. The geometric multiplicity of λ is the number of blocks with λ on the diagonal.

To find the JNF of $A \in \mathbb{R}^{n\times n}$:

1. Compute eigenvalues λ_i and their algebraic multiplicity a_i for A.
2. Find E_{λ_i}, the geometric multiplicities g_i, and eigenvectors $v_{i1}^1, \cdots v_{ig_i}^1$.
3. If $g_i < a_i$, find $a_i - g_i$ missing generalized eigenvectors: $\forall v_{ij}^1 \in \mathbb{C}^n$ associated with λ_i, find all $v_{ij}^k \in \mathbb{C}^n$ s.t. $(A - \lambda_i I)v_{ij}^k = v_{ij}^{k-1}$. (Gaussian elim.)
4. Change of basis $B = [v_{11}^1, \cdots, v_{11}^{k_{11}}, \cdots, v_{1g_1}^1, \cdots, v_{1g_1}^{k_{1g_1}}, \cdots, v_{m1}^1, \cdots, v_{m1}^{k_{m1}}, \cdots, v_{mg_m}^1, \cdots, v_{mg_m}^{k_{mg_m}}]$.
5. Now $J = \begin{bmatrix} J_{k_{11}}(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_{k_{mg_m}}(\lambda_m) \end{bmatrix}$.
6. $B^{-1}AB = J$.

7 Cholesky Decomposition

> $A \in \mathbb{R}^{n\times n}$ is **lower triangular** iff $\forall i < j.A_{ij} = 0$; **upper triangular** iff $\forall i > j.A_{ij} = 0$.
>
> Let $A \in \mathbb{R}^{n\times n}$ be *symmmetric*. If it is *postive definite*, all its diag elems are > 0 and $\forall i, j.\max(A_{ii}, A_{jj}) > |A_{ij}|$. If it is *positive semi-definite*, all its diag elems are ≥ 0 and $\forall i, j.\max(A_{ii}, A_{jj}) \geq |A_{ij}|$. As a consequence, the largest coefficient of A is *on its diagonal*. Also the $1\times 1, \cdots, n\times n$ matrices in the upper left corner of A are also positive (semi-)definite.

The **cholesky decomposition** of $A \in \mathbb{R}^{n\times n}$ is $A = LL^T$ where L is a *lower triangular matrix*. A is *positive semi-definite* $\Leftrightarrow \exists L \in \mathbb{R}^{n\times n}$ s.t. L is lower triangular and $A = LL^T$. A is *positive definite* $\Leftrightarrow \exists L \in \mathbb{R}^{n\times n}$ s.t. L is lower triangular and $A = LL^T$ and $\forall i.L_{ii} > 0$. To find a CD:

1. Consider $L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots \\ l_{21} & l_{22} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$.
2. Solve $A = LL^T$. Example for 3×3:

$$\begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

S is contained in the diagonal elems of L.

8 QR Decomposition

Let $A \in \mathbb{R}^{m\times n} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$ s.t. $a_1, \cdots, a_n$ are linearly independent. Apply gram schmidt to get $(e_1, \cdots, e_n)$ s.t. $\operatorname{span}\{e_1, \cdots, e_n\} = \operatorname{span}\{a_1, \cdots a_n\}$. If we let semi-orthogonal $Q = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \in \mathbb{R}^{m\times n}$ then $A = QR$ with upper triangular:

$$R = \begin{bmatrix} (e_1\cdot a_1) & (e_1\cdot a_2) & \cdots & (e_1\cdot a_n) \\ 0 & (e_2\cdot a_2) & \cdots & (e_2\cdot a_n) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & (e_n\cdot a_n) \end{bmatrix}$$

8.1 Householder Maps

Suppose a hyperplane P goes through $\vec{0}$ with unit normal $u \in \mathbb{R}^m$ ($P = \{x \in \mathbb{R}^m \mid u\cdot x = 0\}$). The **householder matrix** $H_u = I - 2uu^T$ induces reflection wrt P.

9 Convergence in $\mathbb{R}$

Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be a seq of reals and $l \in \mathbb{R}$. (a_n) **converges** to l ($\lim_{n\to\infty} a_n = l$) iff $\forall \epsilon > 0.\exists N \in \mathbb{N}$ s.t. $\forall n > N.|a_n - l| < \epsilon$. (a_n) is a **cauchy seq** iff $\forall \epsilon > 0.\exists N \in \mathbb{N}$ s.t. $\forall n, m > N.|a_n - a_m| < \epsilon$. *$(a_n)$ converges iff it is cauchy.*

9.1 Metric Spaces

A metric space is $\langle S, d\rangle$ where $S \neq \emptyset$ and $d : S\times S \to \mathbb{R}$ satisfies the following:

1. $\forall x, y \in S$, $d(x, y) \geq 0$.
2. $\forall x, y \in S$, $d(x, y) = 0 \Leftrightarrow x = y$.
3. $\forall x, y \in S$, $d(x, y) = d(y, x)$.
4. $\forall x, y, z \in S$, $d(x, y) \leq d(x, z) + d(z, y)$.

If V is a vector space equipped with the norm $\|\cdot\|$, then $\langle V, (x, y) \mapsto \|x - y\|\rangle$ is a metric space.

If $\langle S, d\rangle$ is a metric space and (a_n) is a seq in S, $\lim_{n\to\infty} a_n = l$ iff $\forall \epsilon > 0.\exists N \in \mathbb{N}$ s.t. $\forall n > N.d(a_n, l) < \epsilon$. If (a_n) is converging, the limit is *unique*. (a_n) is **cauchy** iff $\forall \epsilon > 0.\exists N \in \mathbb{N}$ s.t. $\forall n, m > N.d(a_n, a_m) < \epsilon$. (a_n) converges iff it is cauchy.

$\langle S, d\rangle$ is **complete** iff every cauchy seq in S converges in S. For any $k > 0$, $\mathbb{R}^k$ with ℓ_1, ℓ_2 or ℓ_∞ is complete.

9.2 Fixed Point Equations

If $S \neq \emptyset$ and $f : S \to S$, then $p \in S$ is a **fixed point** of f if $f(p) = p$.

$f : S \to S$ is a **contraction** of S in $\langle S, d\rangle$ iff $\exists$ **contraction constant** $0 \leq \alpha < 1$ s.t. $\forall x, y \in S.d(f(x), f(y)) \leq \alpha d(x, y)$. If $\langle S, d\rangle$ is *complete* and f is a *contraction* of S, then *f has a fixed point.*

10 Projectors

> $$\operatorname{proj}_u(v) \triangleq \frac{u\cdot v}{u\cdot u} u$$
>
> The **hermatian conjugate** or **ajoint** of $A \in \mathbb{C}^{m\times n}$ is A^*: A^T with all elems conjugated. **Hermatian Matrix**: $A = A^*$.

P is a **projector** iff $Pv = \operatorname{proj}_u(v)$. Then:

- $P = P^2$ (Reprojection does nothing.)
- $(I - P)$ is also a projector.
- $\operatorname{range}(P) = \operatorname{null}(I - P) \Rightarrow S_1$.
- $\operatorname{range}(I - P) = \operatorname{null}(P) \Rightarrow S_2$.

- If S_1 orthogonal to S_2, then P is an **orthogonal projector** (not orth matrix).

P is an orth proj iff $P = P^*$. If $\hat{Q}$ is an orthonormal basis for range(P), then $P = \hat{Q}\hat{Q}^*$. Also, $P_u = uu^*$. To find orthonormal basis, gram schmidt.

11 Condition Numbers

The measure of the *sensibility* of a problem to small pertrubations in its input. Let problem P take input d and some pertrubation ϵ to give outputs $s(d)$, $s(d+\epsilon)$. Then the **condition number** $\kappa(P)$:

$$\kappa(P) = \max_\epsilon \frac{\|s(d)-s(d+\epsilon)\|}{\|\epsilon\|}$$

The condition number measures the worst case scenario. The relative condition number is defined in terms of relative diff:

$$\kappa(P) = \max_\epsilon \frac{\|s(d)-s(d+\epsilon)\|}{\|\epsilon\|}\frac{\|d\|}{\|s(d)\|}$$

When checking if a system is **stable**, we check the bound of the condition number $\forall d,\epsilon$. If it is bounded, it is stable. A large CN means a **ill-conditioned** problem. A small CN means a **well-conditioned** problem.

11.1 Matrix Condition Numbers

For a non-singular *square* matrix A, its condition number $\kappa(A) = \|A^{-1}\|\|A\|$. If the problem P is represented by the linear equation $Ax = b$, then $\kappa(P) = \kappa(A)$.

The condition number of a *non-square* matrix A is $\kappa(A) = \|A^\dagger\|\|A\|$, where $A^\dagger = (A^TA)^{-1}A^T$ is the **pseudo-inverse** of A.

11.2 Conditioning of a Problem

To decide if a condition number is big enough to say its problem is ill-conditioned, we say: for a condition number κ_0, we lose $\approx \log_{10}(\kappa_0)$ significant figures in accuracy.

12 Stability

Fundamental Theory of Floating Point Arithmetic states all operations have rel error $\leq \epsilon_{\mathtt{machine}}$. Define **problem** $f : \mathcal{X} \to \mathcal{Y}$ and **algorithm** $\tilde{f} : \mathcal{X} \to \mathcal{Y}$. $f(x)$ **well conditioned** if small changes in x mean small changes in $f(x)$. $\tilde{f}$ is **backwards stable** iff $\tilde{f}(x) = f(\tilde{x}) \wedge \frac{\|\tilde{x}-x\|}{\|x\|} = \mathcal{O}(\epsilon_{\mathtt{machine}})$

A backward stable algorithm gives exactly the right answer to nearly the right question. Flops & inner prod backwards stable. Outer prod not. To find if $\tilde{F}$ is:

1. Show $\tilde{f}(x)$ is **exact** for perturbed $\tilde{x}$.
2. Verify rel err in $\tilde{x}$, x is $\propto \epsilon_{\mathtt{machine}}$.

> e.g. *inner product* $x^Ty = \sum_{i=1}^n x_iy_i$:
>
> - $\mathrm{fl}(x^Ty) = \sum_{i=1}^n (x_iy_i)(1+\epsilon_i)$, $|\epsilon_i| \leq \epsilon_{\mathrm{m}}$.
> - $\mathrm{fl}(x^Ty) = \tilde{x}^Ty$, so *exact*.
> - $\frac{|\tilde{x}_i - x_i|}{|x_i|} = \frac{|x_i\epsilon_i|}{|x_i|} = |\epsilon_i| \leq \epsilon_{\mathrm{m}} = \mathcal{O}(\epsilon_{\mathrm{m}})$.

13 Systems of Equations

Basic gaussian elim splits $A = LU$, but this fails if diagonal elems ≈ 0 - not backwards stable. The orders of rows is arbitrary so we can **pivot** using a *orthonormal* **permutation matrix** P (1 non-zero entry per col = 1). PA permutes rows, AP permuts cols. They change $\kappa(A)$. This is stable iff $\rho = \frac{\max_{i,j}|u_{ij}|}{\max_{i,j}|a_{ij}|} = \mathcal{O}(1)$.

13.1 Gradient Optimisation

Func $f : \mathcal{X} \to \mathcal{Z}$ has **level curve** $\mathcal{X} \ni C = \{\vec{x} | f(\vec{x}) = c \in \mathcal{Z}\}$, which can be projected onto $\mathcal{X}$ to visualize f. $\frac{\partial f(\vec{x})}{\partial u} = \begin{bmatrix} f_{x_1} & \cdots & f_{x_m}\end{bmatrix}^T u = \nabla f u$. $\nabla^2 f = H = \begin{bmatrix} f_{x_1x_1} & \cdots & f_{x_1x_m} \\ \vdots & \ddots & \vdots \\ f_{x_mx_1} & \cdots & f_{x_mx_m}\end{bmatrix}$. The grad $\nabla f(\vec{x})$ is the direction of the maximum rate of change $|\nabla f(\vec{x})|$.

Conjugate Gradient Method solves $Ax = b$ by solving $\min_x f(x) = \frac{1}{2}x^TAx - b^Tx$ where $\nabla f(x) = Ax - b$. It chooses *dirs* $p^{(k)}$ s.t. they are conjugate: $\forall k, j. k \neq j \Rightarrow \langle p^{(k)}, p^{(j)}\rangle = p^{(k)T}Ap^{(j)} = 0$ using *residuals* $r^{(k)} = -\nabla f = b - Ax^{(k)}$ and *step size* $\alpha^{(k)}$:

$$p^{(0)} = -\nabla f = b - Ax^{(0)}$$
$$p^{(k)} = r^{(k)} \sum_{i<k} \frac{p^{(i)T}Ar^{(k)}}{p^{(i)T}Ap^{(i)}}p^{(i)}$$

$$\alpha^{(k)} = \arg\min_\alpha f(x^{(k)} + \alpha^{(k)}p^{(k)}) = \frac{p^{(k)T}r^{(k)}}{p^{(k)T}Ap^{(k)}}$$

Converges in $\leq m$ iterations, as $r^{(k)}$s are orthogonal, form a basis for $\mathbb{R}^m$, hence $r^{(m)} = \vec{0}$.

14 Iterative Sol. to Linear Equations

Suppose we wish to solve $Ax = b$ for $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Gaussian elim is $\mathcal{O}(n^3)$, which is impractical. Instead, split $A = G + R$ s.t. $Ax = b \Leftrightarrow x = G^{-1}b - G^{-1}Rx$. Then if $M = -G^{-1}R$ and $c = G^{-1}b$ then $x = Mx + c$. To solve this *iteratively*, we must find a *fixed point* of $f(x) = Mx + c$.

Using *fixed point theorem*, we define seq (x_k) for some starting x_0 s.t. $x_{k+1} = Mx_k + c$.

> Then for a *consistent* norm $\|\cdot\|$, (x_k) converges form any starting point x_0 if $\|M\| < 1$.

The **rate of conv** $r \propto -\log_{10}\|M\|$, so we pick G, R s.t. $-G^{-1}R$ and $G^{-1}b$ are easy to compute, and $\|M\|$ is small.

14.1 Common Splitting

Assume wlog that A has no 0s on the diag (if not so, perform a basis change). Then $A = D + L + U$ where D is the diagonal, and L, U are the strict lower and upper triangular parts of A.

14.2 Jacobi Method

If $A = D + L + U$, we say $A = D + R$ where $R = L + U$, then $Ax = b \Leftrightarrow x = Mx + c$ where $M = -D^{-1}R$ and $c = D^{-1}b$. Once again, consider seq $(x^{(k)})$ defined as $x^{(k+1)} = Mx^{(k)} + c$ and $x^{(0)} \in \mathbb{R}^n$. Then:

$$Mx = -D^{-1}Rx = \begin{bmatrix} -\frac{1}{a_{1,1}}\sum_{j\neq 1} a_{1,j}x_j \\ \vdots \\ -\frac{1}{a_{n,n}}\sum_{j\neq n} a_{n,j}x_j \end{bmatrix}$$

As $x^{(k+1)} = Mx^{(k)} + c$, it comes that:

$$x_i^{(k+1)} = \frac{1}{a_{i,i}}(b_i - \sum_{j\neq i} a_{i,j}x_j^{(k)})$$

So to compute i^{th} element of $x^{(k+1)}$, we only need b, $x^{(k)}$, and the i^{th} row of A, which is great for *parallelization*.

The conv rate is max λ of $D^{-1}R$.

14.3 Gauss-Seidel

Now, we split $A = (D + L) + U$ s.t. $Ax = b \Leftrightarrow x = Mx + c$ where $M = -(D + L)^{-1}U$ and $c = (D + L)^{-1}b$. Once again, consider seq $(x^{(k)})$ defined as $x^{(k+1)} = Mx^{(k)} + c$ and $x^{(0)} \in \mathbb{R}^n$. We can rewrite that as:

$$(D + L)x^{(k+1)} = -Ux^{(k)} + b$$
$$\Updownarrow$$
$$\sum_{j\leq i} a_{i,j}x_j^{(k+1)} = -\sum_{j>i} a_{i,j}x_j^{(k)} + b_i$$

So to compute the i^{th} elem of $x^{(k+1)}$, we only need A, $x^{(k)}$, b and the k^{th} elems of $x^{(k+1)}$ for $k < i$. Since the update is computed with more recent quantities, is faster than jacobi. The conv rate is max λ of $L^{-1}U$.

14.4 Convergence

If $A \in \mathbb{R}^{n\times n}$ is **strictly row diagonally dominant**: $\forall i. |a_{ii}| > \sum_{j\neq i} |a_{ij}|$, then Jacobi and Gauss-Seidel methods *will converge*.

When $\kappa(A)$ is small, conv is faster. If $\kappa(A)$ is too big, it may even diverge.

> A matrix A is **irreducible** if by *symmetric permutations of rows and columns* (i.e. if you swap row a with row b, you must swap col a with col b), it can not take the form $\begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22}\end{bmatrix}$ where $A_{??}$ and $\mathbf{0}$ are block matrices.

If A is *weakly row diagonally dominant* and *irreducible*, Jacobi and G-S still converge.

15 Iteratively Finding Eigenvalues

Suppose we wish to find the eigenvalues and eigenspaces of $A \in \mathbb{R}^{n\times n}$: we must find roots of $\det(A - \lambda I)$ and do gaussian elimination for each eigenvector - an $\mathcal{O}(n^4)$ problem.

15.1 Power Iteration

Let $A \in \mathbb{R}^{n\times n}$ be a diagonalizable matrix with λs of distinct mod $(\forall a, b. |\lambda_a| \neq |\lambda_b|)$. Let $\lambda \in \mathbb{R}$ be **dominant** $(\forall i. |\lambda| \geq |\lambda_i|)$. Consider the seq (x_k) defined by $x_{k+1} = \frac{Ax_k}{\|Ax_k\|}$ and $x_0 \in \mathbb{R}^n \setminus \{0\}$. Then:

$$x_k \underset{k\to\infty}{\longrightarrow} v \qquad \|Ax_k\| \underset{k\to\infty}{\longrightarrow} |\lambda|$$

Where $v \in \mathbb{R}^n$ is the normalized dominant eigenvector. Here, the notion of conv is not rigorous, so we say (x_k) converges to a corresponding eigenspace. However:

- With random x_0 is possible for the it. to give 2nd dominant EVs. *Make sure* there is at least one non-zero component in the corresponding eigenspace.
- For matrix with multiple EVs of max modulus, power it. will converge to a linear combination of the corresponding eigenvectors.
- conv may be slow if the dominant eigenvector is not "very dominant".
- conv `Rate` $\approx \frac{\lambda_2}{\lambda_1}$.

15.2 Inverse Power Iteration

Let $A \in \mathbb{R}^{n\times n}$ be a diagonalizable non-singular matrix with eigenvalues of distinct modulus $(\forall a, b. |\lambda_a| \neq |\lambda_b|)$. This means that $\forall i. \lambda_i \neq 0$ and μ is an eigenvalue of A iff μ^{-1} is an eigenvalue of A^{-1} with the same eigenvectors. So if λ is the smallest eigenvalue of A, λ^{-1} is the dominant eigenvalue of A^{-1}.

Let $\lambda \in \mathbb{R}$ be the eigenvalue of the *smallest modulus*. Consider the seq (x_k) defined by $x_{k+1} = \frac{A^{-1}x_k}{\|A^{-1}x_k\|}$ and $x_0 \in \mathbb{R}^n \setminus \{0\}$. Then:

$$x_k \underset{k\to\infty}{\longrightarrow} v \qquad \|A^{-1}x_k\| \underset{k\to\infty}{\longrightarrow} |\tfrac{1}{\lambda}|$$

Where $v \in \mathbb{R}^n$ is a normalized eigenvector corresponding to λ. Conv rate $\approx \frac{\lambda_{n-1}}{\lambda_n}$.

15.3 Shifts

Let $A \in \mathbb{R}^{n\times n}$ and $s \in \mathbb{R}$. $\lambda \in \mathbb{R}$ is an eigenvalue of A iff $\lambda - s$ is an eigenvalue of **shifted** $A - sI$ with the same eigenvectors. If $A \in \mathbb{R}^{n\times n}$ is a diagonalizable matrix with eigenvalues of distinct modulus, and $A - sI$ is non-singular (s is not an eigenvalue of A), we can find the eigenvalue of A that is closest to s with inverse power iterations on $A - sI$.

15.4 Rayleigh Quotient

For $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n \setminus \{0\}$, the **rayleigh quotient** $R(A, x) = \frac{x^TAx}{x^Tx}$ - an approx of an eigenvalue directly and not its modulus.

15.5 Deflation

To find eigenvalues apart form the dominant, we need to deflate the matrix $A \in \mathbb{R}^{n\times n}$ to $B \in \mathbb{R}^{(n-1)\times(n-1)}$ that has the same eigenvalues as A but the dominant one.

Let $\lambda_1, \cdots, \lambda_n$ by the eigenvalues of A s.t. $|\lambda_1| \geq \cdots \geq |\lambda_n|$ with corresponding eigenvectors $v_1, \cdots, v_n$. Define $H \in \mathbb{R}^{n\times n}$ as a non-singular matrix s.t. $Hx_1 = \alpha e_1$, where $\alpha \in \mathbb{R} \setminus \{0\}$ and $e_1 = [1, 0, \cdots, 0]^T$. Then, we have:

$$HAH^{-1}e_1 = HA\frac{x_1}{\alpha} = H\frac{\lambda_1}{\alpha}x_1 = \lambda_1 e_1$$

Since the 1st col of HAH^{-1} is $[\lambda_1, 0, \cdots, 0]^T$, we can write:

$$HAH^{-1} = \begin{bmatrix} \lambda_1 & b^T \\ \mathbf{0} & B\end{bmatrix}$$

Where $B \in \mathbb{R}^{(n-1)\times(n-1)}$. For any eigenvalue λ and corresponding eigenvector v: $Ax = \lambda x \Leftrightarrow HAx = \lambda Hx \Leftrightarrow HAH^{-1}(Hx) = \lambda Hx$. So λ is an eigenvalue of A iff it is an eigenvalue of HAH^{-1}.

Let λ_2 be the second dominant eigenvector of A s.t. $\lambda_2 \neq \lambda_1$. The corresponding eigenvector $x_2 = H^{-1}\begin{bmatrix}\beta \\ z_2\end{bmatrix}$ with $\beta = \frac{b^Tz_2}{\lambda_2 - \lambda_1}$ and z_2 is the dominant eigenvector of B.

> We assumed that $\lambda_1 \neq \lambda_2$, or that the dominant eigenvalue λ_1 had geometric multiplicity 1. Suppose that instead it had geometric multiplicity p and $\lambda_1 = \cdots = \lambda_p$. Then. deflation works similarly but with blocks instead of vectors: $B \in \mathbb{R}^{(n-p)\times(n-p)}$.

H is the **householder transformation** $H = I - \frac{2uu^T}{u^Tu}$, s.t. $H = H^T = H^{-1}$ is *symmetric* and *orthogonal*. To make sure $Hx_1 = \alpha e_1$, we set: $u = x_1 + \|x_1\|_2 e_1$ and $\alpha = -\|x_1\|_2$.

16 Linear Algebra Basics

- $(ABC)^T = C^TB^TA^T$
- $\vec{u_1} \times \vec{u_2} = \det\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23}\end{vmatrix} = \cdots$
- **Diagonalisable** means all algebraic multiplicities equal to geometric multiplicity.
- **Algmul** - number of times λ appears.
- **Geomul** - number of eigenvectors of λ.
- Eigenvector and eigenval: $Av = \lambda v$
- $\forall u, v \in \mathbb{R}^?. u, v \neq \vec{0} \Rightarrow \mathrm{rk}(uv^T) = 1$.
- **Rank** is the number of eigenvalues $\neq 0$.
- $\det(A^{-1}) = \det(A)^{-1}$.
- $\forall A \in \mathbb{R}^{n\times n}$, complex λ come in pairs with their conjugates $\bar{\lambda}$.
- **Cauchy-Schwarz**: $\forall \vec{x}, \vec{y}. |x \cdot y| \leq \|x\|_2\|y\|_2$
- If A has σ, λ then A^{-1} has $\frac{1}{\sigma}$, $\frac{1}{\lambda}$.

17 Gram Schmidt

The **gram schmidt process** iteratively builds an **orthonormal basis** $(e_1, \cdots, e_n)$ for the n-dimensional subspace generated by $v_j \in \mathbb{R}^m$ for $1 \leq j \leq n$ as follows:

1. To find e_1, let $u_1 = v_1$, let $e_1 = \frac{u_1}{\|u_1\|}$.
2. To find e_2, let $u_2 = v_2 - \mathrm{proj}_{u_1}(v_2) = v_2 - (e_1 \cdot v_2)e_1$. Then $e_2 = \frac{u_2}{\|u_2\|}$.
3. To find e_3, let $u_3 = v_3 - \mathrm{proj}_{u_1}(v_3) - \mathrm{proj}_{u_2}(v_3) = v_3 - (e_1 \cdot v_3)e_1 - (e_2 \cdot v_3)e_2$. Then $e_3 = \frac{u_3}{\|u_3\|}$, and so on $\cdots$.